

# Cauchy $\hat{q}_\psi$ -identity and $\hat{q}_\psi$ -Fermat matrix via $\hat{q}_\psi$ -muting variables of $\hat{q}_\psi$ -Extended Finite Operator Calculus

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## Summary

New  $\psi$ -labeled family of Cauchy identities is found and Fermat matrix notion -now with operator entries - is also  $\psi$ -extended including standard Cauchy and  $q$ -Cauchy or new Fibonomial-Cauchy cases due to the use of  $\hat{q}_\psi$ -commuting variables introduced and promoted as representative for Extended Finite Operator Calculus by the author few years ago.

## 1 I. Towards The Non-commuting

At first we make a  $\psi$ - remark on the notation.  $\psi$  denotes an extension of  $\langle \frac{1}{n!} \rangle_{n \geq 0}$  sequence to quite arbitrary one (the so called - admissible) and the specific choices are for example: Fibonomially-extended ( $\langle F_n \rangle$  - Fibonacci sequence )  $\langle \frac{1}{F_n!} \rangle_{n \geq 0}$  or just "the usual"  $\langle \frac{1}{n!} \rangle_{n \geq 0}$  or Gauss  $q$ -extended  $\langle \frac{1}{nq!} \rangle_{n \geq 0}$  admissible sequences of extended umbral operator calculus. We get used to write these  $q$  - Gauss and other extensions in mnemonic convenient upside down notation [3-6]

$$(1) \quad \psi_n \equiv n_\psi, x_\psi \equiv \psi(x) \equiv \psi_x, n_\psi! = n_\psi(n-1)_\psi!, n > 0,$$

$$(2) \quad x_\psi^k = x_\psi(x-1)_\psi(x-2)_\psi \dots (x-k+1)_\psi$$

$$(3) \quad x_\psi(x-1)_\psi \dots (x-k+1)_\psi = \psi(x)\psi(x-1)_\psi \dots \psi(x-k+1)_\psi.$$

You may consult [3-6] and references therein for further development and use of this notation "q-commuting variables" - included. The idea to use "q-commuting variables" goes back at least to Cigler (1979) [1] ( see formula (7) , (11) in [1] ) and also Kirchenhofer - see [2] for further systematic development ) We shall take here notation from [3-6] and the results from [3-6] - for granted- in view of an easy access via ArXiv to the source papers . For other respective references to Cigler, Kirchenhofer and Others see: [3-6]. The task is to invent what to replace with the question mark in the Cauchy identity type formula (4) below

$$(4) \quad \sum_{k \geq 0} (?) \binom{r}{k}_\psi \binom{s}{j-k}_\psi = \binom{r+s}{j}_\psi, \binom{n}{k}_\psi = 0, \text{ for } k < 0.$$

to get true new extended identities . The simple minded way to do it is closed because for

$$(x +_\psi 1)^r (x +_\psi 1)^s = \left( \sum_{k \geq 0} \binom{r}{k}_\psi x^k \right) \left( \sum_{l \geq 0} \binom{s}{l}_\psi x^l \right)$$

$$E(\partial_\psi) x^{r+s} \equiv (x +_\psi 1)^{r+s} \equiv \sum_{j \geq 0} \binom{r+s}{j}_\psi x^j$$

the regular arrive at the Cauchy identity fails as

$$(x + \psi y)^r (x + \psi y)^s \neq (x + \psi y)^{r+s}.$$

For example:

$$(x +_F y)^2 = x^2 + F_2 xy + y^2, (x +_F y)^4 = x^4 + F_4 x^3 y + F_4 F_3 x^2 y^2 + F_4 x y^3 + y^4.$$

$$(x +_F y)^5 = x^5 + F_5 x^4 y + F_5 F_4 x^3 y^2 + \dots + F_5 x y^4 + y^5.$$

$$(x +_F y)(x +_F y)^4 \neq (x +_F y)^5.$$

where  $\{F_n\}_{n>0}$  denotes the Fibonacci sequence (see subsequent pages). What then would help perhaps [3-6] is to replace commuting variables  $xy = yx$  by non-commuting ones as done by Cigler (see: also Kirichenhofer) in Umbral Calculus domain. The idea to use "q-commuting variables" goes back at least to Cigler (1979) [1,2] (see formula (7), (11) in [1]) and also Kirichenhofer - see [2] for further systematic development). In [2] Kirichenhofer equivalently defined the polynomial sequence of  $q$ -binomial type by [5,4]

$$(5) \quad p_n(A+B) \equiv \sum_{k \geq 0} \binom{r}{k}_q p_k(A) p_{n-k}(B) \text{ where } [B, A]_q \equiv BA - qAB = 0.$$

$A$  and  $B$  might be interpreted here as co-ordinates on quantum  $q$ -plane (see [7] Chapter 4). For example  $A = \hat{x}$  and  $B = y\hat{Q}$  where  $\hat{Q}\varphi(x) = \varphi(qx)$ , (more on that [5,4]).

In the  $q$ -case (see: Proposition 4.2.3 in [7]- with  $yx = qxy$ ) we have

$$(6) \quad \sum_{k \geq 0} q^{(r-k)(j-k)} \binom{r}{k}_q \binom{s}{j-k}_q = \binom{r+s}{j}_q$$

hence from this Cauchy  $q$ -identity here we define immediately ( $s = i, s = j$ ) the symmetric  $q$ -Pascal (or  $q$ -Fermat) matrix elements via the following easy to find out formula:

$$(7) \quad \sum_{k \geq 0} q^{(r-k)(j-k)} \binom{i}{k}_q \binom{j}{k}_q = \binom{i+j}{j}_q.$$

For arbitrary admissible sequences labeling the elements of the giant family of Extended Finite Operator Calculus - **calculi** of Rota, Roman and Others the present author introduced in [4,5] appropriate (also for our purpose here) notions -to be used next. Note: **Calculi** means "stones" (or "pebbles" or "counters") in Latin. And really indeed - some choices of admissible  $\psi$ -sequences are impressive combinatorial.

## 2 II. Examples of choices and $\hat{q}_\psi$ -binomial symbol

Let since now on  $\psi = \{\psi_n(q)\}_{n \geq 0}$  as in [3-6]. With the Gauss choice  $\psi_n(q) = [n_q!]^{-1}$  where  $n_q = \frac{1-q^n}{1-q}$  and  $n_q! = n_q(n-1)_q!, 1_q! = 0_q! = 1$  we may interpret  $q$ -binomial coefficients in a standard way, namely:  $q$ -Gaussian coefficient  $\binom{n}{k}_q$  denote number of  $k$ -dimensional subspaces in  $n$ -th dimensional space over Galois field  $GF(q)$  [8,9] i.e. we are dealing with lattice of subspaces. For  $q = 1$  we arrive at the lattice of

subsets and the binomial coefficient standard interpretation. In [8] the combinatorial interpretation was proposed also for Fibonomial coefficients

$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} \equiv \frac{n_F^k}{k_F!}, \quad n_F \equiv F_n \neq 0,$$

where we make an analogy driven [6,4,3] identifications ( $n > 0$ ):

$$n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \dots 2_F 1_F;$$

$$0_F! = 1; \quad n_F^k = n_F(n-1)_F \dots (n-k+1)_F.$$

(Here:  $\psi_n(q) = [F_n!]^{-1}$ ). In [8,9] a partial ordered set was defined in such a way that the Fibonomial coefficients count the number of specific finite "birth-self-similar" sub-posets of this infinite non-tree poset naturally related to the Fibonacci tree of rabbits growth process. For fascinating "weighted choices" - Konvalina combinatorial interpretations see [10,11]. In order to proceed we take from [4,5] only this what we need now. (Vector spaces are over the field of zero characteristics). An so let us define the main notion of this note [4,5].

**Definition 1** Let  $\{p_n\}_{n \geq 0}$  be the  $\partial_\psi$ -basic polynomial sequence of the  $\partial_\psi$ -delta operator  $Q(\partial_\psi) = Q$ . Then the  $\hat{q}_{\psi,Q}$ -operator is a linear map

$$\hat{q}_{\psi,Q} : P \rightarrow P; \quad \hat{q}_{\psi,Q} p_n = \frac{(n+1)_\psi - 1}{n_\psi} p_n, \quad n \geq 0.$$

We call this useful  $\hat{q}_{\psi,Q}$  operator the  $\hat{q}_{\psi,Q}$ -mutator operator.

**Note:** For  $Q = id$   $Q(\partial_\psi) = \partial_\psi$  the natural notation is  $\hat{q}_{\psi,id} \equiv \hat{q}_\psi$ . For  $Q = id$  and  $\psi_n(q) = \frac{1}{R(q^n)!}$  and  $R(x) = \frac{1-x}{1-q}$   $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,id} \equiv \hat{q}_R \equiv \hat{q}_{q,id} \equiv \hat{q}_q \equiv \hat{q}$  and  $\hat{q}x^n = q^n x^n$ .

**Definition 2** Let  $A$  and  $B$  be linear operators acting on  $P$ ;  $A : P \rightarrow P$ ,  $B : P \rightarrow P$ . Then  $AB - \hat{q}_{\psi,Q}BA \equiv [A, B]_{\hat{q}_{\psi,Q}}$  is called  $\hat{q}_{\psi,Q}$ -mutator of  $A$  and  $B$  operators.

Consider then the following special case  $\hat{q}_\psi$  of linear on  $P = F[x]$   $\hat{q}_{\psi,Q}$ -mutator operator :

$$\hat{q}_\psi : P \rightarrow P; \quad \hat{q}_\psi x^n = \frac{(n+1)_\psi - 1}{n_\psi} x^n; \quad n \geq 0.$$

Consider also  $\hat{q}_\psi$ -muting variables  $yx = \hat{q}_\psi xy$ . Introduce also a  $\hat{q}$ -binomial symbol:

**Definition 3** We define  $\hat{q}_\psi$ -binomial symbol i.e.  $\hat{q}_\psi$ -Gaussian coefficients as follows:

$$\binom{n}{k}_{\hat{q}_\psi} = \frac{n_{\hat{q}_\psi}!}{k_{\hat{q}_\psi}!(n-k)_{\hat{q}_\psi}!} = \binom{n}{n-k}_{\hat{q}_\psi} \quad \text{where} \quad n_{\hat{q}_\psi}! = n_{\hat{q}_\psi}(n-1)_{\hat{q}_\psi}! , 1_{\hat{q}_\psi}! = 0_{\hat{q}_\psi}! = 1$$

and  $n_{\hat{q}_\psi} = \frac{1-\hat{q}_\psi^n}{1-\hat{q}_\psi}$  for  $n > 0$ .

**Challenge 1** Are we facing possibility of  $\hat{q}_\psi$ -quantum "groups" investment alike [7] ?

### 3 III. $\psi$ -sequences labeled family of Cauchy identities

Equipped with the above we discover immediately the being looked for Cauchy  $\psi$ -identity formula and the  $\psi$ - extended Fermat matrix (including  $q$ - Fermat and Fibonomial  $F$ -Fermat matrix cases). Namely let us observe that the following is true.

### Observation 1

$$p_n(A+B) \equiv \sum_{k \geq 0} \binom{n}{k}_{\hat{q}_\psi} p_k(A) p_{n-k}(B)$$

where

$$[B, A]_{\hat{q}_\psi} \equiv BA - \hat{q}_\psi AB = 0$$

and in particular

$$(x+y)^n = \sum_{k \geq 0} \binom{n}{k}_{\hat{q}_\psi} x^k y^{n-k}$$

where

$$[y, x]_{\hat{q}_\psi} = 0.$$

**Challenge 2** Are we (compare with [5]) facing the possibility of systematic representation of a general umbral calculus in Rota-like operator form [3-6] with help of  $\hat{q}_\psi$ -quantum "plane" variables in place of "q-commuting variables" employed by Cigler (1979) [1] and Kirichenhofer - to do their splendid efficient job - this time in

$\hat{q}_\psi$ -extended binomial enumeration (this refers to fundamental binomial enumeration formulation of umbra in [12])? - Are we - ...?

In the  $\hat{q}_{\psi,Q}$ -case of the  $\hat{q}_{\psi,Q}$  from [5,6] general extended umbral theory in Rota-like finite operator form we have

### Observation 2

$$(8) \quad \sum_{k \geq 0} \hat{q}_\psi^{(r-k)(j-k)} \binom{r}{k}_{\hat{q}_\psi} \binom{s}{j-k}_{\hat{q}_\psi} = \binom{r+s}{j}_{\hat{q}_\psi}$$

From Observation 2 i.e. from the Cauchy  $\hat{q}_\psi$  - identity we infer the following  $\hat{q}_\psi$ -formula for matrix elements of the symmetric  $\hat{q}_\psi$ -Pascal (or  $\hat{q}_\psi$ -Fermat) matrix elements

$$(9) \quad \sum_{k \geq 0} \hat{q}_\psi^{(r-k)(j-k)} \binom{i}{k}_{\hat{q}_\psi} \binom{j}{k}_{\hat{q}_\psi} = \binom{i+j}{j}_{\hat{q}_\psi}.$$

For the first most recent applications see [13]. For  $q$ -Pascal matrix see [14].

In analogy to the standard case [15-17,14] we shall call the matrices (compare with [14,13]) - with operator valued matrix elements

$$x^{i-j} \binom{i}{j}_{\hat{q}_\psi}$$

and

$$\binom{i+j}{j}_{\hat{q}_\psi}$$

the  $\hat{q}_\psi$ -Pascal  $P[x]$  and  $\hat{q}_\psi$ -Fermat  $F[1]$  matrices - correspondingly.

The result from [15-17] tempt to be  $\hat{q}_\psi$ -extended.

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